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Effective Action for the Quarks in Instanton–Vacuum Model

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Abstract

Effective action for the quarks is discussed in the model of instanton–based vacuum consisting of a superposition of instanton anti–instanton fluctuations. The case $N_f = 1$ is considered. Comparison with another approach is carried out.

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Recently there was essential progress in understanding the mechanism of the chiral symmetry spontaneous breaking (CSSB)[1–4]. As a consequence of this, the physics of pseudoscalar octet particles become more transparent[3,5]. The instanton "fluid" vacuum model allowed us to describe low–energy characteristics of (π, k) mesons in a good agreement with experimental data[3,5,6].

The general method for calculating the correlation functions in instanton medium proposed by Dyakonor and Petrov is the following. The instanton anti–instanton superpositions must be considered as an external classical field. The correlators in the presence of this field will depend on the characteristics of all pseudoparticles, i.e., their dimensions ρ, orientation U, and center Z. The averaging over statistical ensemble of instantons finally give exact correlator in instanton vacuum. In this case, the averaging is substantially simplified due to the following approximations.

1. The packing parameter of the instanton fluid $\bar{\rho}/R$, when R is the mean distance between pseudoparticles, is small $\bar{\rho}/R \simeq \frac{1}{3}$[2], and therefore the pseudoparticles may be considered uncorrelated.

2. When a number of colors N_c is large, the instanton distribution as a function of instanton size ρ is very narrow and tends to the δ–shaped function at $N_c \rightarrow \infty$, so that in the leading order over $1/N_c$ all sizes of instantons ρ_I may be replaced by their mean values $\bar{\rho}$[2].

3. Hilbert space of the fermions truncate to the space of zero modes. This approximation is justified for the long wavelength properties of the vacuum (scales > 0.3 fm). The reason is that the spectrum of the Dirac operator in the field of an instanton is characterized by a gap of about $\bar{\rho}^{-1} \simeq 600$ MeV between the zero energy state and the continuum of scattering states[3,6,8].

Diagrammatic procedure for averaging was developed and two point correlators were calculated. However, three and higher–point functions are not obtained by such a method. The procedure ladder diagrams calculation in different directions does not exist. In general, the solution of the coupled Bethe–Salpether type equations is technically difficult to obtain.

Therefore a problem of finding a reliable algorithm for calculation of another Green function arises. Besides that, it is interesting to find the effective action which is equivalent to the previous model. There are several attempts to write an action of such type[4,6,7,8]. Nevertheless, all of these actions are not equivalent to the previous model, (see, for example, [6]). Moreover, these attempts lead to Green functions which are singular on quark mass as it will be shown below.

Let us begin with QCD partition function.

$$Z = \int \mathcal{D}\psi \mathcal{D}\psi^+ \mathcal{D}A_\mu \;\; \ell^{S_\psi + S_{gl}} \tag{1}$$

The main points of the model are following integration on gauge field A_μ is equivalent to averaging over the statistical ensemble of pseudoparticles with known distribution function of given configuration and to replacing field A_μ in the interaction part by superposition of instanton anti-instanton configuration.

$$\begin{aligned} Z' &= \int \mathcal{D}\psi \mathcal{D}\psi^+ \;\; << exp\left[\int \psi^+ \left(i\hat{\partial} + i\hat{A} + im\right) \psi d^4x + \eta^+\psi + \psi^+\eta\right] >> \\ A &= \sum_I A_I + \sum_{\bar{I}} A_{\bar{I}} \end{aligned} \tag{2}$$

Where << ... >> denotes the average over the collective coordinates: positions of all pseudoparticles z, their orientations in color space U_I/following to approximation 2 we will substitute all sizes of pseudoparticles by $\bar{\rho} = (600 \text{ HeV})^{-1}$. However, note that Z' is not similar to the former model since in definition (2) the propagator has the following form

$$S' = \frac{<< \mathcal{D}et\left(i\hat{\nabla} + im\right)\left(i\hat{\partial} + i\hat{A} + im\right)^{-1} >>}{<< \mathcal{D}et\left(i\hat{\nabla} + im\right) >>} \tag{3}$$

The denominator here also depends on pseudoparticle characteristics and cannot be averaged independently, i.e., propagator (3) is not equal to one which was obtained in [3].

$$\mathcal{S} = << -\left(i\hat{\partial} + i\hat{A} + im\right)^{-1} >> \tag{4}$$

Therefore, in our opinion, if we wish to construct a theory which would be similar to (4), it is necessary to start from

$$Z\left(\eta, \eta^{+}\right) = \int \mathcal{D}\psi \mathcal{D}\psi^{+} << \frac{e^{\int \psi^{+}\left(i\hat{\nabla}+im\right)\psi d^{4}x + \eta^{+}\psi + \psi^{+}\eta}}{det\left(i\hat{\nabla} + im\right)} >> \tag{5}$$

Following 3, we will replace the exact Green functions in the field of one instanton by a model one[3]

$$S_I(x,y) = S_0(x,y) - \frac{\Phi_0^I(x)\Phi_0^{I+}(y)}{im} \tag{6}$$

where $S_0(x,y)$ is the free propagator, $\Phi_0^I(x)$ is the zero mode for I-th pseudoparticle: it is a right (left)-handed Weyl spinor for the (anti) instanton. This model is exact at small momenta $\bar{\rho}p << 1$ and at large momenta $\bar{\rho}p >> 1$. At small momenta (the range important to the CSSB) the exact propagator coincides with the singular part of (6). Using

$$S_0^{-1} = -(i\hat{\partial} + im) \text{ and } S_I^{-1} - S_0^{-1} = -i\hat{A}_I$$

we have

$$-i\hat{A} = \sum_I \left(S_I^{-1} - S_0^{-1}\right) + \sum_{\bar{I}} \left(S_{\bar{I}}^{-1} - S_0^{-1}\right)$$

$$\mathcal{S} = \left[S_0^{-1} + \sum_I \left(S_I^{-1} - S_0^{-1}\right) + \sum_{\bar{I}} \left(S_{\bar{I}}^{-1} - S_0^{-1}\right)\right]^{-1} \tag{7}$$

From (6) and (7) we have

$$i\hat{\nabla}+im = i\hat{\partial}+im+\left\{\sum_I \left[\left(1 + \left(i\hat{\partial} + im\right)\frac{\Phi_I\Phi_I^{+}}{im}\right)^{-1} - 1\right]\left(i\hat{\partial} + im\right)\right\}+\left\{I \to \bar{I}\right\} \tag{8}$$

And for Z from (5) we obtain

$$Z\left(\eta,\eta^+\right) = \int \mathcal{D}\psi\mathcal{D}\psi^+ << \frac{[\,\ell^{\int \psi^+ i\hat{\partial}\psi d^4x + \eta^+\psi + \psi^+\eta}]\prod_I \left(1 - \frac{1}{im}\int \psi^+ i\hat{\partial}\Phi_I d^4x \int \Phi_I^+ i\hat{\partial}\psi d^4x'\right)\prod_I (I \to I)}{det\left(i\hat{\nabla}\right)} >> \tag{9}$$

Here we use: a) the only "opposite–charged" pseudoparticles enter into overlap–integral $< \Phi_I^+|i\hat{\partial}|\Phi_I >= \delta_{II}, b)\psi, \psi^+$ are Grassmannia variable c) mass matrix in S_0 is set to zero (since we are interested here by the chiral limit).

Using (8) and doing the trick, similar to the one used by Ilgenfritz [9], we rewrite the $\left[det(i\hat{\partial} + i\hat{A})\right]^{-1}$ in eq. (9) in terms of scalar spinor fields χ.

$$\begin{aligned} det^{-1}\left(i\hat{\partial} + i\hat{A}\right) &= const \int \mathcal{D}\chi\mathcal{D}\chi^+ \ell^{\int \chi^+ i\hat{\partial}\chi d^4y} \\ &\prod_I \left[exp\left(-\frac{1}{im}\right)\int \chi^+ i\hat{\partial}\Phi_I d^4x \int \Phi_I^+ i\hat{\partial}\chi d^4x'\right]\prod_I \left[I \to \bar{I}\right] \end{aligned} \tag{10}$$

And for Z we have now

$$Z\left(\eta,\eta^+\right)_{N_f = 1} = const \int \mathcal{D}\psi\mathcal{D}\psi^+\mathcal{D}\chi\mathcal{D}\chi^+ \ell^{\int(\psi^+ i\hat{\partial}\psi + \chi^+ i\hat{\partial}\chi)d^4x + \eta^+\psi + \psi^+\eta}$$

$$<< \prod_I \left\{1 - \frac{1}{im} < \psi^+|i\hat{\partial}|\Phi_I >< \Phi_I^+|i\hat{\partial}|\psi > \ell^{-\frac{1}{im}<\chi^+|i\hat{\partial}|\Phi_I><\Phi_I^+|i\hat{\partial}|\chi>}\right\}\prod_I \left\{I \to \bar{I}\right\} >> \tag{11}$$

Now the theory with Z' can be obtained from Z, if one "drop out" the variables χ, χ^+.

$$\begin{aligned} Z'(\eta,\eta^+) &= const \int \mathcal{D}\psi\mathcal{D}\psi^+ \ \ell^{\int \psi^+ i\hat{\partial}\psi d^8x + \eta\psi^+ + \psi\eta^+} \\ &\prod_I \left(1 - \frac{1}{im} < \psi^+|i\hat{\partial}|\Phi_I >< \Phi_I^+|i\hat{\partial}\psi >\right)\prod_I \left(I \to \bar{I}\right) \end{aligned} \tag{12}$$

This expression (12) differs from the result of [4] since (12) is normalized not by m as in [4], but by finite quantity as in [7]. Let us now investigate the problem of how far we may normalize the theory by m, putting $m = 0$ from the very beginning. Let us calculate (12) with nonzero m, and then lend m to zero. After averaging over positions and the orientations of pseudoparticles, using the density matrix composed of zero fermion modes[3].

$$\Phi_{i\alpha}(x)\Phi^{+}_{j\beta}(x') = \int \frac{d^4k_1 d^4k_2}{(2\pi)^8} \ell^{i(k_1,(x-z))-i(k_2,(x'-z))} \frac{\phi(k_1)\phi(k_2)}{8|k_1|k_2|}$$

$$\left(\hat{k}_1\gamma_\mu\gamma_\nu\hat{k}_2\frac{1-\gamma_5}{2}\right) ij \quad \left(U\tau^{-}_{\mu}\tau^{+}_{\nu}U^{+}\right) \alpha\beta$$

Here $\alpha,\beta(i,j)$ are color (spinor) indices, $\tau^{\pm}_{\mu}$ are $N_c \times N_c$ matrices with $(\bar{\tau} \pm i)$ standing in the left upper corner (all other elements are zero) τ–are Pauli matrices, $\hat{k} = k_\mu\gamma_\mu$. The function $\phi(k)$ connects with Fourier–transformed zero modes.

$$\phi(k) = \pi\bar{\rho}^2 \frac{d}{dz} \left[I_0(z)K_0(z) - I_1(z)K_1(z)\right]_{z=\frac{|k|\bar{\rho}}{2}} = \begin{cases} -\frac{2\pi\bar{\rho}}{|k|} & k\bar{\rho} \ll 1 \\ -\frac{12\pi}{k^4\bar{\rho}^2} & k\bar{\rho} \gg 1 \end{cases} \tag{13}$$

For anti–instanton $\gamma_5 \to -\gamma_5, \tau^{\pm}_{\mu} \to \tau^{\mp}_{\mu}$, when averaging over the orientations with the Haare measure normalized to unity we use the relations

$$\int dU = 1, \quad \int dU\ U^{\alpha}_{\beta}U^{\gamma}_{\delta} = \frac{1}{N_c}\ \delta^{\alpha}_{\delta}\delta^{\alpha}_{\beta}$$

$$\int dU \quad U^{\alpha_1}_{\delta_1}U^{+\gamma_1}_{\beta_1}U^{\alpha_2}_{\delta_2}U^{+\gamma_2}_{\beta_2} = \frac{1}{N_c^2}\ \delta^{\alpha_1}_{\beta_1}\delta^{\gamma_1}_{\delta_1}\delta^{\alpha_2}_{\beta_2}\delta^{\gamma_2}_{\delta_2} + \frac{4}{N_c^2-1}\ (\lambda^{\alpha})^{\alpha_1}_{\beta_1}(\lambda^{\alpha})^{\alpha_2}_{\beta_2}(\lambda^{\beta})^{\gamma_1}_{\delta_1}(\lambda^{\beta})^{\gamma_2}_{\delta_2} \tag{14}$$

λ's are the generators of $SU(N_c)$. We can obtain

$$Z' = \text{ const } \int \mathcal{D}\psi\mathcal{D}\psi^{+} \quad \ell^{\int \psi^{+}i\hat{\partial}\psi d^4x+\eta\psi^{+}+\psi\eta^{+}}(Y_{+})^{N_{+}}(Y_{-})^{N_{-}}$$

where $Y_{\pm} = 1 - \frac{1}{imVN_c}\int \frac{d^4k}{(2k)^4}\psi^{+}_{L(R)}(k)a^2(k)\psi_{L(R)}(k), N_{+(-)}-$ is the number of instantons (anti–instantons, $a(k) = |k|\phi(k), \psi_{L(R)} = \frac{1\pm\lambda_5}{2}\psi$. Then in thermodynamic limit $V \to \infty N_{\pm} \to \infty\frac{N_{\pm}}{V} =$ const using well–known equation.

$$\lim_{N\to\infty}\left(1+\frac{x}{N}\right)^N = e^x$$

we obtain

$$Z'\left(\eta,\eta^{+}\right) = \text{const} \int \mathcal{D}\psi\mathcal{D}\psi^{+}\, \ell^{\int \psi^{+}\left(-\hat{k}+\frac{iN}{2VN_c}\frac{\phi^2 k^2}{m}\right)\psi\frac{d^4k}{(2k)^4}+\eta\psi^{+}+\psi^{+}\eta}$$

It means that $M(k) \sim \frac{1}{m}$ and $\lim m \to 0$ does not exist. Theory is singular! If we put $m = 0$ in the case when Z is normalized by m then we obtain uncertainty $\left(\frac{0}{0}\right)$ for any Green functions. Let us now return to Z. Integrate first the (11) over the orientation matrices U_I.

Denote the integrals under consideration by $I_0, (I_1)_{ij}, \lambda = \frac{i}{m}$

$$I_0 = \int \exp\left\{\lambda\int\left(\chi^{+}i\hat{\partial}\Phi_I\right)d^4x \int\left(\Phi_I^{+}i\hat{\partial}\chi\right)d^4y\right\}dU$$

$$(I_1)_{ij} = \int \left(i\hat{\partial}\Phi\right)_i\left(\Phi_I^{+}i\hat{\partial}\right);\exp\left\{\lambda\int\left(\chi^{+}i\hat{\partial}\Phi_I\right)d^4x\int\left(\Phi_I^{+}i\hat{\partial}\chi\right)d^4y\right\}dU$$

In leading order on N_c using (13) and the observation [10] that the group integration is equivalent to the projection of a tensor product of fundamental representations onto to the singlets of the group (example eq. (14) see [6]) we can obtain that

$$\begin{aligned} I_o &= \frac{1}{1-\frac{\lambda}{N_c}\left(\chi_L^{+}\chi_L\right)} \\ \left(\chi_L^{+}\chi_L\right) &\equiv \int \chi_L^{+}(k_1)\chi_L(\ell_1)\ell^{-z_I(k_1-e_1)}d\Gamma_1 \\ d\Gamma_1 &= d\Gamma(k_1,\ell_1) = \frac{a(k_1)\,a(\ell_1)\,d^4k_1d^4\ell_1}{(2\pi)^8} \end{aligned} \tag{15}$$

For anti-instanton $(L \to R)$. Note that I_0 and I_1 are connected

$$\lambda(I_1)_{ij} + \lambda^2\frac{\partial(I)_{ij}}{\partial\lambda} = \frac{\partial^2 I_o}{\partial\chi_i^{+}\partial\chi_j}$$

and to resolve this differential equation we find

$$\begin{aligned} I_{1ij} = &\frac{\lambda/N_c}{1-\lambda/N_c(\chi_L^{+}\chi_L)}\left\{\int\left(\frac{1+\gamma_5}{2}\right)_{ij}\ell^{-z_I i(k-\ell)}\ell^{ikx-i\ell y}d\Gamma(k,\ell) + \right. \\ &+ \left.\frac{\lambda}{N_c}\int \chi_{Li}(\ell_1)\chi_{Lj}^{+}(k)\ell^{-z_I(k+k_1-\ell-\ell_1)}\ell^{ik_1x-i\ell y}\frac{d\Gamma(k,\ell)d\Gamma(k_1,\ell_1)}{1-\frac{\lambda}{N_c}(\chi_L^{+}\chi_L)}\right\} \end{aligned} \tag{16}$$

Substituting (15), (16) and (11) and integrating over positions of pseudoparticles $\left(\frac{d^4 z_I}{V(4)}\right)$ we obtain

$$
\begin{aligned}
Z\left(\eta,\eta^+\right) &= \text{const} \int \mathcal{D}\psi\mathcal{D}\psi^+\mathcal{D}\chi\mathcal{D}\chi^+ \\
&\qquad \ell^{\int\left(\psi^+ i\partial\psi + \chi_i^+\partial\chi\right)d^4x + \eta\chi^+ + \psi^+\eta} \left\{\frac{1}{V}\int d^4 z K_+\right\}^{N+} \left\{\frac{1}{V}\int d^4 z K_-\right\}^{N}_{-}
\end{aligned}
$$

$$
\begin{aligned}
K_+ &= \frac{1}{1-\frac{i}{mN_c}\left(\chi_L^+\chi_L\right)} + \frac{\frac{i}{mN_c}\int \psi_L^+(k_1)\psi_L(\ell_1)\ell^{-iz(k_1-\ell_1)}d\Gamma_1}{1-\frac{i}{mN_c}\left(\chi_L^+\chi_L\right)} \\
&+ \frac{\left(\frac{i}{mN_c}\right)^2\int\left(\psi_L^+(k_1)\psi_L(\ell_1)\right)\left(\chi_L^+(k_2)\psi_L(\ell_2)\right)\ell^{-Z(k_1+k_2-\ell_1-\ell_2)}d\Gamma_1 d\Gamma_2}{\left(1-\frac{i}{mN_c}\left(\chi_L^+\chi_L\right)\right)^2} \\
K_- &= K_+(L\to R) \qquad\qquad (17)
\end{aligned}
$$

This expression is finite over m and we can use chiral limit here. Besides, for propagator calculation we can restrict ourselves by $\frac{\bar\rho}{R}$ approximations. (The quark momenta involved in the pion physics appear to be parametrically less than $1/\bar\rho$.[3])

$$
\begin{aligned}
&\int d^4k d^4q \ell^{i(k-q)x}\chi^+_{L(R)}(k)\chi_{L(R)}(q)a(k)a(q) \\
&\simeq a^2(0)\int_0^{1/\rho}\chi^+_{L(R)}(k)k^2dk^2d\Omega\int_0^{1/\rho}\chi_{L(R)}(q)q^2dq^2d\Omega \equiv C_{+(-)}
\end{aligned}
$$

The values of $C_{+(-)}$ are fixed from integration by χ,χ^+. Then from (17) we obtain

$$
\begin{aligned}
Z(0,0) &= \text{const}\int\mathcal{D}\psi\mathcal{D}\psi^+\ell^{\int\psi^+ i\hat\partial\psi d^4x}\left(\mathcal{L}_+\right)^{N_+}\left(\mathcal{L}_-\right)^{N_-} \\
&\quad \int\mathcal{D}\chi\mathcal{D}\chi^+\ell^{\int\chi^+ i\hat\partial\chi d^4x - N_+\ell n C_+ - N_-\ell n C_-} \\
&\quad \delta\left(C_+-\left(\chi_L^+\chi_L\right)\right)\delta\left(C_--\left(\chi_R^+\chi_R\right)\right)dC_+dC_- \\
\mathcal{L}_+ &= \left\{\frac{1}{V}\int\psi_L^+(k)ia^2(k)\psi_L(k)d^4k\right\} \quad \mathcal{L}_- = \mathcal{L}_+\left(N_+\to N_-, L\to R\right) \qquad (18)
\end{aligned}
$$

After integration over $\chi, \chi^+, C_\pm$, we obtain $\Big($using the replacement

$$\int \mathcal{D}\psi\mathcal{D}\psi^+ F(\psi,\psi^+) \equiv \int \mathcal{D}\psi\mathcal{D}\psi^+ \int \frac{d\beta_+}{2\pi} \int \frac{d\beta_-}{2\pi} \int d\Gamma_+ d\Gamma_- \ .$$

$$. \ \ell^{\left[\int d^4x\psi^+ i\hat{\partial}\psi + i\beta_+(\mathcal{L}_+ - \Gamma_+) + N_+ \ell n\Gamma_+ + i\beta_-(\mathcal{L}_- - \Gamma_-) + N_- \ell n\Gamma_-\right]}\Big)$$

$$\begin{aligned} Z_{N_f=1}(0,0) &= \frac{\text{const}}{\det(i\hat{\partial})} \int d\beta_+ d\beta_- \exp(-N_+\ell n\beta_+ - N_-\ell n\beta_-) \int \mathcal{D}\psi\mathcal{D}\psi^+ \\ & \exp\psi^+(k)\left[-\hat{k} + ia^2(k)\left(\beta_+\left(\frac{1+\gamma_5}{2}\right) + \beta_-\left(\frac{1-\gamma_5}{2}\right)\right)\right]\psi(k) \end{aligned} \tag{19}$$

This result was first obtained in [4]. Integration in (19) on $\beta_\pm$ can be performed by the saddle point method, and in thermodynamical limit we can obtain $|N_+ = N_- = \frac{N}{2}|$ for $N_f = 1$ a theory of non–interesting quarks with the dynamical mass $M(k)$[4,6,7]

$$\begin{aligned} Z_{QCD}(N_f = 1) &= \text{const} \int \mathcal{D}\psi\mathcal{D}\psi^+ \exp\left[\int \frac{d^4k}{(2\pi)^4}\psi^+(k)\left(-\hat{k} + iM(k)\right)\psi(k)\right] \\ M(k) &= \frac{\epsilon N}{2VN_c}k^2\phi^2(k) \end{aligned} \tag{20}$$

Quantity ϵ here is determined from the gap equation derived by Dyakonov and Petrov[3], which naturally emerges when integrated with $\beta_\pm$.

$$\frac{4V^{(4)}N_c}{N}\int \frac{d^4k}{(2\pi)^4}\frac{M^2(k)}{M^2(k)+k^2} = 1 \tag{21}$$

Recalling that $N/V^{(4)} \sim$ (200 MeV) one obtains from (21) $M(0) \simeq 300$ MeV and $< \bar{\psi}\psi > \simeq (-250MeV)^3$[3].

Certainly our progress in comparison with the results of [4,6,7] is still not substantial. Nevertheless it seems to us that our approach to the problem under consideration is conceptionally more correct and derivation of results for multiple–point Green functions from (17) is rather a technical problem which we hope to resolve in forthcoming publications.

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